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SUBOPTIMAL UNIVERSAL STRATEGIES IN AN OPTIMUM-TIME GAME PROBLEM†

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A differential game in which the payoff functional is the time required for the phase point to reach the target set is considered. A construction of ε-optimal strategies, similar to the standard construction when the value function is everywhere differentiable, is proposed. The difference is that the gradient of a non-smooth and discontinuous value function is replaced by a certain quasi-gradient.

This paper continues the investigation of universal strategies in [1–4]. Certain facts from the theory of generalized solutions of first-order partial differential equations [5–7] and from non-smooth analysis [8] are used. The quasi-gradient of the value function is defined instead of the gradient in the standard construction of optimal strategies, which assumes that the value function is everywhere differentiable. An analogous construction for a differential game with fixed stopping time and continuous value function was considered in [9].

1. Let the motion of the controlled system be described by the equation

$$\dot{x}(t) = f(x(t), p(t), q(t)), \quad t \geqslant 0 \tag{1.1}$$

where $x(t) \in R^n$ is the phase state of the system at time t, $p(t) \in P$ and $q(t) \in Q$ are the controls of players I and II, respectively, and $P \subset R^l$ and $Q \subset R^m$ are compact sets. It is assumed that the function $f(x, p, q)$ is jointly continuous in its variables and satisfies a Lipschitz condition with respect to x

$$\| f(x+y, p, q) - f(x, p, q) \| \leqslant \lambda \| y \| \tag{1.2}$$

for all $(x, p, q) \in R^n \times P \times Q$. It is also assumed that

$$\min_{p \in P} \max_{q \in Q} \langle s, f(x, p, q) \rangle = \max_{q \in Q} \min_{p \in P} \langle s, f(x, p, q) \rangle = H(x, s) \tag{1.3}$$

for any $s \in R^n$ and $x \in R^n$.

Let $M \subset R^n$ be a given set in phase space. Player I tries to ensure that the phase point $x(t)$ will reach M in the least possible time. Player II, for his part, tries either to prevent the encounter with M or to maximize the time to the encounter. Different versions of the rigorous formulation of these problems are known and the existence of a game equilibrium has been proved. In this paper we will use the formalization of positional differential games [10].

Positional strategies of players I and II are arbitrary functions

$$\mathbf{R}^n \ni x \mapsto U(x) \in P, \; \mathbf{R}^n \ni x \mapsto V(x) \in Q$$

respectively. Suppose that player I has chosen a certain strategy U and a partition

$$\Delta = \{0 = t_0 < t_1 < \ldots\}, \quad \lim_{i \to \infty} t_i = \infty$$

If $x_0 \in R^n$ is a given point, the symbol $X(x_0, U, \Delta)$ will denote the set of trajectories $x(\cdot)$: $[0, \infty) \mapsto R^n$ of the differential inclusion

$$\dot{x}(t) \in \operatorname{co}\{f(x(t), U(x(t_i)), q) : q \in Q\}, \quad t \in [t_i, t_{i+1}), \; t_i \in \Delta, \; x(0) = x_0$$

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Similarly, suppose that player II has chosen a strategy V and a partition Δ. The symbol $X(x_0, V, \Delta)$ will denote the set of solutions of the differential inclusion

$$\dot{x}(t) \in \text{co}\{f(x(t),\ p, V(x(t_i))): p \in P\},\quad t \in [t_i, t_{i+1}),\ t_i \in \Delta,\ x(0) = x_0$$

Define a functional

$$\tau_\varepsilon(x(\cdot)) := \min\{t \in \mathbf{R}^+ : x(t) \in M^\varepsilon\}$$

If $x(t) \notin M^\varepsilon$ for all $t \in R^+$, we define $\tau_\varepsilon(x(\cdot)) = \infty$. Here ε is a positive number and M^ε is the ε-neighbourhood of the set M, i.e.

$$M^\varepsilon := \{x + y : x \in M,\ \|y\| \leqslant \varepsilon\}$$

We shall use the notation

$$\text{diam}(\Delta) := \sup_i (t_{i+1} - t_1) \text{ for } i = 0, 1, 2 \ldots$$

We know (see, for example, [10]) that for any initial position $x_0 \in R^n$ the game has a value Val (x_0) $\in [0, \infty]$, i.e. the following conditions hold:

1. for any numbers $\theta <$ Val (x_0) and $\varepsilon > 0$, player I has a strategy U such that

$$\lim_{\text{diam}(\Delta)\downarrow 0} \sup \sup\{\tau_\varepsilon(x(\cdot)) : x(\cdot) \in X(x_0, U, \Delta)\} \leqslant \theta$$

2. for any number $\theta <$ Val (x_0) a number $\varepsilon > 0$ and a strategy V for player II exist such that

$$\lim_{\text{diam}(\Delta)\downarrow 0} \inf \inf\{\tau_\varepsilon(x(\cdot)) : x(\cdot) \in X(x_0, V, \Delta)\} \geqslant \theta$$

The existence of the value has been proved [10] for a differential game in the class of strategies $U(t, x)$, $V(t, x)$, which depend on the variable t both in the case of the controlled system $\dot{x} = f(t, x, p, q)$ and in the case of a stationary system of type (1.1). The strategies considered in this paper will not depend on t. We also note that the strategies constructed below have a universality property: they guarantee ε-optimal solutions from any initial position in a bounded domain.

2. Consider the following boundary-value problem for the Isaacs–Bellman equations

$$H(x, D\upsilon(x)) + 1 = 0,\quad x \in G \tag{2.1}$$

$$\upsilon(x) = 0,\quad x \in \partial G \tag{2.2}$$

where $H(x, s)$ is the Hamiltonian defined by (1.3); $G = R^n \backslash M$ is an open domain, $\bar{G}$ is the closure of G and ∂G is the boundary of G.

We recall the following result [11]. Let $\upsilon: \bar{G} \mapsto R^+$ be a continuous function that satisfies the boundary condition (2.2), is continuously differentiable in G and satisfies Eq. (2.1) in that domain. Then the function υ is identical with the value of the differential game.

Moreover, in that case optimal strategies U_0 and V_0 for the two players may be constructed as follows. We introduce extremal pre-strategies

$$p_0(x, s) \in \text{Arg}\min_{p\in P}\left[\max_{q\in Q}\langle s, f(x, p, q)\rangle\right] \tag{2.3}$$

$$q_0(x, s) \in \text{Arg}\max_{q\in Q}\left[\min_{p\in P}\langle s, f(x, p, q)\rangle\right] \tag{2.4}$$

We define the strategies U_0 and V_0 are superpositions of the pre-strategies and the gradient $D\upsilon$, i.e.

$$U_0(x) := p_0(x, D\upsilon(x)),\quad V_0(x) := q_0(x, D\upsilon(x)) \tag{2.5}$$

The assumption that the value function is smooth holds only in exceptionally rare cases. The value function may be discontinuous and may take the improper value $+\infty$. In the general case, however, as shown below, one can define ε-optimal strategies by formulae of type (2.5), provided the gradient $D\upsilon(x)$ is replaced by a certain quasi-gradient. In the construction proposed here, we will use results obtained [5, 7] for bounded solutions of Dirichlet-type problems for first-order partial differential equations.

Consider the transformation [12]

$$[0,\infty] \ni \upsilon \mapsto u(\upsilon) = 1 \pm e^{\pm\upsilon} \in [0,1]$$

It is obvious that the function $\upsilon(x)$ satisfies (2.1) if and only if the function $u(x) = 1 - \exp(-\upsilon(x))$ satisfies the equation

$$H(x, Du(x)) + 1 - u(x) = 0 \tag{2.6}$$

It was shown in [7] that Eq. (2.6) has a generalized (minimax) solution u: $\widetilde{G} \mapsto [0, 1]$ satisfying the boundary condition

$$u(x) = 0, \quad x \in \partial G \tag{2.7}$$

and that solution is unique. The minimax solution u is lower semicontinuous and possesses the following property.

Let $\eta \in G$, $q_* \in Q$, $\tau > 0$. Let $Y(\eta, q_*)$ be the set of trajectories $y(\cdot)$: $[0, \tau] \mapsto R^n$ of the differential inclusion

$$\dot{y}(t) \in \operatorname{co}\{f(y(t), p, q_*): p \in P\} \tag{2.8}$$

with initial condition $y(0) = \eta$. Assume that $y(t) \in G$ for all $y(\cdot) \in Y(\eta, q_*)$ and all $t \in [0, \tau]$. Then a trajectory $y(\cdot) \in Y(\eta, q_*)$ exists such that

$$(u(\eta) - 1)e^{\tau} \geqslant u(y(\tau)) - 1 \tag{2.9}$$

This property is equivalent to the u-stability condition for the function υ [10] and to the definition of an upper solution of Eq. (2.6).

Note that the value function is related to the minimax solution of problem (2.6), (2.7) by the equality

$$\operatorname{Val}(x) = \pm\ln(1 \pm u(x)), \quad x \in \overline{G} \tag{2.10}$$

Suboptimal strategies for the players may be defined as superpositions of pre-strategies and quasi-gradients of the minimax solution. As corollaries of these constructions one can demonstrate the existence of the value and prove equality (2.10).

3. We will now describe the construction of an ε-optimal strategy for player I. Let u be the minimax solution of problem (2.6), (2.7). Define

$$u_\alpha(x) := \min_{y \in \overline{G}} [u(y) + w_\alpha(x, y)] \tag{3.1}$$

where

$$w_\alpha(x, y) = \frac{\left(\alpha^{2/\nu} + \|x - y\|^2\right)^\nu}{\alpha}, \quad \nu = \frac{1}{2 + 2\lambda}, \quad 0 < \alpha < \min\left\{\frac{1}{3}, \frac{1}{\lambda(1+\lambda)}\right\} \tag{3.2}$$

λ being the Lipschitz constant (see 1.2)). The function w_α satisfies the inequality

$$H(x, D_x w_\alpha(x, y)) - H(y, -D_y w_\alpha(x, y)) - w_\alpha(x, y) \leqslant 0 \tag{3.3}$$

for any $(x, y) \in G \times G$ such that $\|x - y\| \leqslant 1$. Functions of this type are used to prove uniqueness theorems in the theory of generalized solutions of first-order partial differential equations (see, for example, [7, 6]).

Choose any point

$$y_\alpha(x) \in \operatorname{Arg}\min_{y\in G}[u(y) + w_\alpha(x,y)] \tag{3.4}$$

Such a point exists because u is lower semicontinuous.

It can be shown that

$$u_\alpha(x) \leqslant u(x) + \alpha, \quad \|x - y_\alpha(x)\| \leqslant 2\alpha \tag{3.5}$$

Define

$$U_\alpha(x) = p_0(x, s_\alpha(x)) \tag{3.6}$$

where p_0 is the extremal pre-strategy defined by condition (2.3)

$$s_\alpha(x) := (\mathrm{D}_x w_\alpha)(x, y_\alpha(x)) = -(\mathrm{D}_y w_\alpha)(x, y_\alpha(x)) \tag{3.7}$$

If u is continuously differentiable in the neighbourhood of a point $x \in G$, it follows from (3.4) that

$$\mathrm{D}u(y_\alpha(x)) + (D_y w_\alpha)(x, y_\alpha(x)) = 0$$

By (3.7) and (3.5), we obtain $s_\alpha(x) \to \mathrm{D}u(x)$ as $\alpha \to 0$. We may therefore call $s_\alpha(x)$ the quasi-gradient of u at x. Referring to (2.10), we see that $U_\alpha(x) = p_0(x, s_\alpha^b(x))$, where $s_\alpha^b(x) = s_\alpha(x)(1 - u(x))^{-1}$ is the quasi-gradient of the value function Val (x) in the sense described above.

Theorem 1. Let $u\colon \overline{G} \mapsto [0, 1]$ be the minimax solution of problem (2.6), (2.7). Let D be a compact subset of G. Assume that

$$\theta^0 = \sup_{x\in D}[-\ln(1 - u(x))] < \infty$$

Then for any $\varepsilon > 0$ one can choose a parameter value $\alpha > 0$ so that, for any point $x_0 \in D$

$$\lim_{\operatorname{diam}(\Delta)\downarrow 0} \sup \sup\{\tau_\varepsilon(x(\cdot)) : x(\cdot) \in X(x_0, U_\alpha, \Delta)\} \leqslant -\ln(1 - u(x_0)) + \varepsilon \tag{3.8}$$

where U_α is a strategy of type (3.6).

Proof. Let $X(x_0)$ denote the set of trajectories $x(\cdot)\colon R^+ \mapsto [0, \infty)$ of the differential inclusion

$$\dot{x}(t) \in \operatorname{co}\{f(x(t), p, q) : p \in P,\ q \in Q\}$$

satisfying the initial condition $x(0) = x_0$. Put

$$\begin{aligned} K &:= \{x(t) \in R^n : x(\cdot) \in X(x_0),\ t \in [0, \theta^0 + \varepsilon],\ x_0 \in D\} \\ m &:= \sup\{\|f(x + h, p, q)\| : x \in K,\quad p \in P,\ q \in Q,\ \|h\| \leqslant 1\} \end{aligned} \tag{3.9}$$

Choose numbers $\alpha > 0$ and $\delta_0 > 0$ so that

$$3\alpha < \varepsilon, \quad \delta_0 m \leqslant \alpha, \quad 3\alpha < 1 \tag{3.10}$$

Choose $x_0 \in D$ arbitrarily.

We shall prove the following proposition. Let $x(\cdot) \in X(x_0, U_\alpha, \Delta)$, $t_i \in \Delta$, $t_i < \theta = -\ln(1 - u(x_0)) + \varepsilon$ and let $\operatorname{dist}(x(t_i); M) > 3\alpha$. Then for any $\tau \in [t_i, t_{i+1}] \cap [0, \theta]$

$$u_\alpha(x(\tau)) \leqslant 1 - [1 - u_\alpha(x(t_i))]e^{\tau - t_i} + (\tau - t_i)e^{\tau - t_i}h(\alpha, \delta) \tag{3.11}$$

where $\delta = \operatorname{diam}(\Delta)$, $\lim_{\alpha\to 0}\lim_{\delta\to 0} h(\alpha, \delta) = 0$. The quantity $h(\alpha, \delta)$ depends only on α and δ but not on the choice of the point $x_0 \in D$ and the trajectory $x(\cdot) \in X(x_0, U_\alpha, \Delta)$.

In addition to (3.10), we assume that the parameters α and δ have been chosen in such a way that the following estimate holds

$$e^{\theta}[\alpha + \theta h(\alpha, \delta)] < e^{\varepsilon} - 1 \tag{3.12}$$

Suppose that these estimates hold. Given $x(\cdot) \in X(x_0, U_\alpha, \Delta)$, let us consider two cases: (1) a time $t_i \in \Delta$ exists such that $t_i < \theta$ and dist $(x(t_i); M) \leqslant 3\alpha$; (2) the inverse inequality dist $(x(t_i); M) \leqslant 3\alpha$ holds for all $t_i \in \Delta$ that satisfy the estimate $t_i < \theta$.

Since $3\alpha \leqslant \varepsilon$, it follows that in case 1

$$\tau_\varepsilon(x(\cdot)) \leqslant t_i \leqslant \theta = -\ln(1 - u(x_0)) + \varepsilon \tag{3.13}$$

Now consider case 2. The recurrent estimates (3.11) yield

$$u_\alpha(x(\theta)) \leqslant 1 - [1 - u_\alpha(x_0)]e^{\theta} + \theta e^{\theta} h(\alpha, \delta)$$

Note that $e^{\theta} = e^{\varepsilon}(1 - u(x_0))^{-1}$. By (3.5), $u_\alpha(x_0) \leqslant u(x_0) + \alpha$. Consequently

$$u_\alpha(x(\theta)) \leqslant 1 - e^{\varepsilon} + e^{\theta}[\alpha + \theta h(\alpha, \delta)] < 0$$

The last inequality follows from (3.12). Thus, in case 2 we have derived the inequality $u_\alpha(x(\theta)) < 0$. On the other hand, by (3.1) we have $u_\alpha(x(\theta)) \geqslant 0$. This contradiction proves that case 2 is impossible.

Thus, if the estimate (3.11) is true, the strategy U_α satisfies the estimate (3.8). Hence the proof of the theorem reduces to verifying (3.11).

Let us introduce the notation

$$\xi = x(t_i),\ \ \eta = y_\alpha(\xi),\ \ s_* = s_\alpha(\xi),\ \ p^* = U_\alpha(\xi) = p_0(\xi, s_*),\ \ q_* = q_0(\eta, s_*) \tag{3.14}$$

(recall that p_0 and q_0 are the pre-strategies defined by (2.3) and (2.4)). Since the functions $f(x, p, q)$ and $U_\alpha(x)$ are independent of t, we may put $t_i = 0$. Define a vector

$$f^* = \frac{x(\tau) - \xi}{\tau} = \frac{1}{\tau}\int_0^{\tau} \dot{x}(t)dt,\ \ \dot{x}(t) \in \mathrm{co}\{f(x(t), p^*, q): q \in Q\} \tag{3.15}$$

It is required to prove the inequality

$$1 - u_\alpha(\xi) \leqslant e^{-\tau}[1 - u_\alpha(\xi + f^*\tau)] + \tau h(\alpha, \tau) \tag{3.16}$$

It follows from the inequality dist $(\xi; M) > 3\alpha$ and the second estimate in (3.5) that dist $(\eta, M) > \alpha$. We recall that $\delta_0 m \leqslant \alpha$. It follows from this estimate and the definition of m (3.9) that $y(t) \notin M$ for any trajectory of the differential inclusion (2.8) and any $t \in [0, \tau]$. It follows from (2.9) that

$$1 - u(\eta) \leqslant e^{-\tau}[1 - u(\eta + f_*\tau)] \tag{3.17}$$

where

$$f_* = \frac{y(\tau) - \eta}{\tau} = \frac{1}{\tau}\int_0^{\tau} \dot{y}(t)dt,\ \ \dot{y}(t) \in \mathrm{co}\{f(y(t), p, q_*): p \in P\} \tag{3.18}$$

We recall that $\eta = y_\alpha(\xi)$, so that (3.1) and (3.4) give $u_\alpha(\xi) = u(\eta) + w_\alpha(\xi, \eta)$. Combining this estimate with (3.17), we obtain

$$1 - u_\alpha(\xi) \leqslant e^{-\tau}[1 - u(\eta + f_*\tau)] - w_\alpha(\xi, \eta)$$

Let us add and subtract the expression $e^{-\tau}w_\alpha(\xi + f^*\tau, \eta + f_*\tau)$ on the right of this inequality. By definition (3.1), we have

$$u_\alpha(\xi + f^*\tau) \leqslant u(\eta + f_*\tau) + w_\alpha(\xi + f^*\tau, \eta + f_*\tau)$$

Consequently

$$1-u_\alpha(\xi)\leqslant e^{-\tau}[1-u_\alpha(\xi+f^*\tau)]+\Delta_\alpha$$
$$\Delta_\alpha=e^{-\tau}w_\alpha(\xi+f^*\tau,\eta+f_*\tau)-w_\alpha(\xi,\eta) \tag{3.19}$$

Let us estimate Δ_α. The function w_α is continuously differentiable. By (3.7) and (3.14) we have $s_* = D_x w_\alpha(\xi,\eta) = -D_y w_\alpha(\xi,\eta)$. Therefore

$$\Delta_\alpha\leqslant e^{-\tau}\left[w_\alpha(\xi,\eta)+\left\langle s_*,f^*\right\rangle\tau-\left\langle s_*,f_*\right\rangle\tau\right]-w_\alpha(\xi,\eta)+h_1(\alpha,\tau)\tau\leqslant$$
$$\leqslant\left[\left\langle s_*,f^*\right\rangle-\left\langle s_*,f_*\right\rangle-w_\alpha(\xi,\eta)+h_2(\alpha,\tau)\right]\tau$$

where, as below, $h_i(\alpha,\tau)\to 0$ as $\tau\to 0$. These quantities depend only on τ but not on the specific motion $x(\cdot)\in X(x_0, U_\alpha, \Delta)$.

It follows from (2.3) and (3.14) that

$$\left\langle s_*,f\right\rangle\leqslant H(\xi,s_*)\quad\forall f\in\operatorname{co}\{f(\xi,p^*,q):q\in Q\}$$

Consequently, we have the following inequality for the vector f^* of (3.15)

$$\left\langle s_*,f^*\right\rangle\leqslant H(\xi,s_*)+h_3(\alpha,\tau)$$

Similarly, it follows from (2.4), (3.14) and (3.18) that

$$\left\langle s_*,f_*\right\rangle\geqslant H(\eta,s_*)-h_4(\alpha,\tau)$$

The estimates just established yield

$$\Delta_\alpha\leqslant[H(\xi,s_*)-H(\eta,s_*)-w_\alpha(\xi,\eta)+h_5(\alpha,\tau)]\tau$$

Using (3.3), we get

$$\Delta_\alpha\leqslant h_5(\alpha,\tau)\tau$$

Substituting this inequality into (3.19), we obtain the required estimate (3.16), where $h(\alpha,\tau) = h_5(\alpha,\tau)$, $\lim_{\alpha\to 0}\lim_{\tau\to 0} h_5(\alpha,\tau) = 0$. It is obvious from the above estimates that the quantity h_5 may be defined so that it does not depend on the choice of $x(\cdot)\in X(x_0, U_\alpha, \Delta)$.

This completes the proof of the theorem.

4. There is an analogous construction of an ε-optimal strategy for player II. Suppose we are given an initial point $x_0\in G$ and a number $\theta < -\ln(1-u(x_0))$. As before, u is the minimax solution of problem (2.6), (2.7). It follows from the definition of this solution [5, 7] that a sequence of lower solutions u_k exists such that $0\leqslant u_k(x_0)\leqslant u(x_0)\leqslant 1$ and $\lim_{k\to\infty} u_k(x_0) = u(x_0)$. One can therefore choose a lower solution u_* such that

$$\theta<-\ln(1-u_*(x_0))$$

The function u_* is upper semicontinuous and possesses the following property: for any $\eta\in G$, $p_*\in P$ and $\tau > 0$, a trajectory $y(\cdot)\colon [0,\tau]\mapsto R^n$ of the differential inclusion

$$\dot y(t)\in\operatorname{co}\{f(y(t),p_*,q):q\in Q\}$$

exists which satisfies the initial condition $y(0) = \eta$, such that

$$(u(\eta)-1)e^\tau\leqslant u(y(\tau))-1$$

Put

$$u_*^\alpha(x):=\max_{y\in G}[u_*(y)-w_\alpha(x,y)]$$

where the function w_α is defined, as before, by (3.2). Choose a point

$$y^{\alpha}(x) \in \operatorname{Arg}\max_{y \in G}[u_*(y) - w_{\alpha}(x, y)]$$

We define a strategy V_α: $\overline{G} \mapsto Q$ for player II by

$$V_{\alpha}(x) = q_0(x, s^{\alpha}(x))$$

$$s_{\alpha}(x) := -(D_x w_{\alpha})(x, y^{\alpha}(x)) = (D_y w_{\alpha})(x, y^{\alpha}(x))$$

where q_0 is a pre-strategy of type (2.4). The following proposition is true for the strategies V_α.

Theorem 2. Given an initial point $x_0 \in G$ and starting time $\theta < -\ln(1 - u(x_0))$, one can choose parameters $\alpha > 0$ and $\varepsilon > 0$ so that the following estimate holds

$$\lim_{\operatorname{diam}(\Delta)\downarrow 0} \inf \inf\{\tau_{\varepsilon}(x(\cdot)) : x(\cdot) \in X(x_0, V_{\alpha}, \Delta)\} \geqslant \theta$$

The proof is basically the same as that of Theorem 1.

Note that we have derived an estimate for the guaranteed result with the initial point fixed. The construction of V_α may be adjusted in such a way that the estimate will hold for all points x_0 in a given compact set D.

Theorems 1 and 2 imply Eq. (2.10). If a sufficiently small parameter α is chosen, the strategies U_α and V_α guarantee players I and II results that are as close to optimal as desired.

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